

A CHARACTERIZATION OF THE DISCONTINUITY POINT SET OF STRONGLY SEPARATELY CONTINUOUS FUNCTIONS ON PRODUCTS

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ABSTRACT. We study properties of strongly separately continuous mappings defined on subsets of products of topological spaces equipped with the topology of pointwise convergence. In particular, we give a necessary and sufficient condition for a strongly separately continuous mapping to be continuous on a product of an arbitrary family of topological spaces. Moreover, we characterize the discontinuity point set of strongly separately continuous function defined on a subset of countable product of finite-dimensional normed spaces.

1. INTRODUCTION

In 1998 Omar Dzagnidze [1] introduced a notion of a strongly separately continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}$. Namely, he calls a function f *strongly separately continuous* at a point $x^0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$ if for every $k = 1, \dots, n$ the equality $\lim_{x \rightarrow x^0} |f(x_1, \dots, x_k, \dots, x_n) - f(x_1, \dots, x_k^0, \dots, x_n)| = 0$ holds. Dzagnidze proved that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is strongly separately continuous at x^0 if and only if f is continuous at x^0 .

Extending the investigations of Dzagnidze, J. Činčura, T. Šalát and T. Visnyai in [2] consider strongly separately continuous functions defined on the space ℓ_2 of sequences $x = (x_n)_{n=1}^\infty$ of real numbers such that $\sum_{n=1}^\infty x_n^2 < +\infty$ endowed with the standard metric $d(x, y) = (\sum_{n=1}^\infty (x_n - y_n)^2)^{1/2}$. In particular, the authors gave an example of a strongly separately continuous everywhere discontinuous function $f : \ell_2 \rightarrow \mathbb{R}$.

Recently, T. Visnyai in [3] continued to study properties of strongly separately continuous functions on ℓ_2 and constructed a strongly separately continuous

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function $f : \ell_2 \rightarrow \mathbb{R}$ which belongs to the third Baire class and is not quasi-continuous at every point. Moreover, T. Visnyai gave a sufficient condition for strongly separately continuous function to be continuous on ℓ_2 .

In this paper we study strongly separately continuous functions defined on subsets of a product $\prod_{t \in T} X_t$ of topological spaces X_t equipped with the Tychonoff topology of pointwise convergence. In Section 2 we introduce a notion and give the simplest properties of \mathcal{S} -topology which is tightly connected with strongly separately continuous functions. In the third section we give a necessary and sufficient condition for a strongly separately continuous mapping to be continuous on a product of an arbitrary family of topological spaces. In Section 4 we find a necessary condition on a set to be the discontinuity point set of a strongly separately continuous mapping defined on a product of topological spaces. Finally, in the fifth section we describe the discontinuity point set of a strongly separately continuous function defined on a subset of a countable product of finite-dimensional normed spaces.

2. A NOTION AND PROPERTIES OF \mathcal{S} -TOPOLOGY

Let $X = \prod_{t \in T} X_t$ be a product of a family of sets X_t with $|X_t| > 1$ for all $t \in T$. If $S \subseteq S_1 \subseteq T$, $a = (a_t)_{t \in T} \in X$, $x = (x_t)_{t \in S_1} \in \prod_{t \in S_1} X_t$, then we denote by a_S^x a point $(y_t)_{t \in T}$, where

$$y_t = \begin{cases} x_t, & t \in S, \\ a_t, & t \in T \setminus S. \end{cases}$$

In the case $S = \{s\}$ we shall write a_s^x instead of $a_{\{s\}}^x$.

If $n \in \mathbb{N}$, then

$$\sigma_n(x) = \{y = (y_t)_{t \in T} \in X : |\{t \in T : y_t \neq x_t\}| \leq n\}$$

and

$$\sigma(x) = \bigcup_{n=1}^{\infty} \sigma_n(x).$$

Each of the sets of the form $\sigma(x)$ for an $x \in X$ is called a σ -product of X .

We denote by τ the Tychonoff topology on a product $X = \prod_{t \in T} X_t$ of topological spaces X_t . If $X_0 \subseteq X$, then the symbol (X_0, τ) means the subspace X_0 equipped with the Tychonoff topology induced from (X, τ) .

If $X_t = X$ for all $t \in T$ then the product $\prod_{t \in T} X_t$ we also denote by $X^{\mathfrak{m}}$, where $\mathfrak{m} = |T|$.

Definition 2.1. A set $A \subseteq \prod_{t \in T} X_t$ is called \mathcal{S} -open if $\sigma_1(x) \subseteq A$ for all $x \in A$.

We remark that the definition of an \mathcal{S} -open set develops the definition of a set of the type (P_1) introduced in [2].

Let $\mathcal{S}(X)$ denote the collection of all \mathcal{S} -open subsets of X . We notice that $\mathcal{S}(X)$ is a topology on X . We will denote by (X, \mathcal{S}) the product $X = \prod_{t \in T} X_t$ equipped with the topology $\mathcal{S}(X)$.

The proof of the following is straightforward.

Proposition 2.1. *Let $X = \prod_{t \in T} X_t$, $|X_t| > 1$ for all $t \in T$, $x \in X$ and $A \subseteq X$.*

Then

- (1) $A \in \mathcal{S}(X)$ if and only if $X \setminus A \in \mathcal{S}(X)$;
- (2) $A \in \mathcal{S}(X)$ if and only if $A = \bigcup_{x \in A} \sigma(x)$;
- (3) $\sigma(x)$ is the smallest \mathcal{S} -open set which contains x ;
- (4) if $A \in \mathcal{S}(X)$, then A is dense in (X, τ) .
- (5) there exists a non-trivial \mathcal{S} -open subset of X if and only if $|T| \geq \aleph_0$.

It follows from Proposition 2.1 that σ -products of two distinct points of $\prod_{t \in T} X_t$ either coincide, or does not intersect. Consequently, the family of all σ -products of an arbitrary \mathcal{S} -open set $X \subseteq \prod_{t \in T} X_t$ generates a partition of X on mutually disjoint \mathcal{S} -open sets, which we will call \mathcal{S} -components of X .

Notice that every \mathcal{S} -component of X is an indiscrete subspace of $(X, \mathcal{S}(X))$ and the space $(X, \mathcal{S}(X))$ is a topological sum of indiscrete spaces.

Proposition 2.1 easily implies the following properties of the \mathcal{S} -topology.

Proposition 2.2. *Let $X = \prod_{t \in T} X_t$ and $|X_t| > 1$ for all $t \in T$. Then the space (X, \mathcal{S})*

- (1) *is first-countable;*
- (2) *does not satisfy T_0 , T_1 and T_2 if $T \neq \emptyset$;*
- (3) *satisfy T_3 , $T_{3\frac{1}{2}}$ and T_4 ;*
- (4) *is arcwise connected if and only if $|T| < \aleph_0$;*
- (5) *is extremally disconnected.*

3. A NECESSARY AND SUFFICIENT CONDITION FOR A STRONGLY SEPARATELY CONTINUOUS MAPPING TO BE CONTINUOUS

Definition 3.1. Let $(X_t : t \in T)$ be a family of topological spaces, Y be a topological space and let $X \subseteq \prod_{t \in T} X_t$ be an \mathcal{S} -open set. A mapping $f : X \rightarrow Y$

is said to be *separately continuous at a point* $a = (a_t)_{t \in T} \in X$ *with respect to the t -th variable* provided that the mapping $g : X_t \rightarrow Y$ defined by the rule $g(x) = f(a_t^x)$ for all $x \in X_t$ is continuous at the point $a_t \in X_t$.

Let $f : X \rightarrow Y$ be a mapping between topological spaces X and Y , $a \in X$ and $b \in Y$. Then $\lim_{x \rightarrow a} f(x) = b$ if and only if for any neighborhood V of b in Y there is a neighborhood U of a in X such that $f(U) \subseteq V$.

Definition 3.2. Let $X \subseteq \prod_{t \in T} X_t$ be an \mathcal{S} -open set, \mathcal{T} be a topology on X and let (Y, d) be a metric space. A mapping $f : (X, \mathcal{T}) \rightarrow Y$ is called *strongly separately continuous at a point $a \in X$ with respect to the t -th variable* if

$$\lim_{x \rightarrow a} d(f(x), f(x_t^a)) = 0.$$

Definition 3.3. A mapping $f : X \rightarrow Y$ is

- *(strongly) separately continuous at a point $a \in X$* if f is (strongly) separately continuous at a with respect to each variable $t \in T$;
- *(strongly) separately continuous on the set X* if f is (strongly) separately continuous at each point $a \in X$ with respect to each variable $t \in T$.

Definition 3.4. A set $A \subseteq \prod_{t \in T} X_t$ is called *projectively symmetric with respect to a point $a \in A$* if for all $t \in T$ and for all $x \in A$ we have $x_t^a \in A$.

Definition 3.5. Let $X \subseteq \prod_{t \in T} X_t$ and \mathcal{T} be a topology on X . Then (X, \mathcal{T}) is said to be *locally projectively symmetric* if every $x \in X$ has a base of projectively symmetric neighborhoods with respect to x .

It is easy to see that an arbitrary \mathcal{S} -open subset of a product $\prod_{t \in T} X_t$ of topological spaces X_t equipped either with the Tychonoff topology τ , or with the \mathcal{S} -topology, is a locally projectively symmetric space. All classical spaces of sequences as the space c of all convergence sequences or the spaces ℓ_p with $0 < p \leq \infty$ are locally projectively symmetric.

Proposition 3.1. Let X be an \mathcal{S} -open subset of a product $\prod_{t \in T} X_t$ of topological spaces equipped with a locally projectively symmetric topology \mathcal{T} , $a = (a_t)_{t \in T} \in X$, Y be a metric space and let $f : (X, \mathcal{T}) \rightarrow Y$ be a continuous mapping at the point a . Then f is strongly separately continuous at a .

Proof. Fix $\varepsilon > 0$ and $t \in T$. Take a projectively symmetric with respect to a neighborhood U of a such that

$$d(f(x), f(a)) < \frac{\varepsilon}{2}$$

for all $x \in U$. Then $x_t^a \in U$ and

$$d(f(x), f(x_t^a)) \leq d(f(x), f(a)) + d(f(a), f(x_t^a)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $x \in U$. □

Note that the inverse implication is not true in case (X, \mathcal{T}) is not a locally projectively symmetric space, as the example below shows.

Example 1. Let X be the Niemytski plane, i.e. $X = \mathbb{R} \times [0, +\infty)$, where a base of neighborhoods of $(x, y) \in X$ with $y > 0$ form open balls with the center in (x, y) , and a base of neighborhoods of $(x, 0)$ form the sets $U \cup \{(x, 0)\}$, where U is an open ball which tangents to $\mathbb{R} \times \{0\}$ in the point $(x, 0)$. Then there exists a continuous function $f : X \rightarrow \mathbb{R}$, which is not strongly separately continuous.

Proof. Denote $a = (0, 0)$ and consider an arbitrary basic neighborhood U of a in X . Since X is a completely regular space, there exists a continuous function $f : X \rightarrow [0, 1]$ such that $f(a) = 0$ and $f(x, y) = 1$ for all $(x, y) \in X \setminus U$. We show that f is not strongly separately continuous at the point a with respect to the second variable. Indeed, fix $\varepsilon = \frac{1}{2}$ and a neighborhood V of a . By the continuity of f at a we choose a neighborhood V_0 of a such that $V_0 \subseteq V$ and $f(x, y) < \frac{1}{2}$ for all $(x, y) \in V_0$. Now let $(x, y) \in V_0$ be an arbitrary point with $x \neq 0$. Then $f(x, 0) = 1$ and $|f(x, y) - f(x, 0)| = 1 - f(x, y) > \frac{1}{2}$. Hence, f is not strongly separately continuous. \square

Let \mathcal{E} denote the Euclidean topology on \mathbb{R} , $X_1 = X_2 = (\mathbb{R}, \mathcal{E})$ and let \mathcal{T} be a discrete topology on the set $X = \mathbb{R} \times \mathbb{R}$. We consider the following function $f : X \rightarrow \mathbb{R}$,

$$f(x_1, x_2) = \begin{cases} 1, & (x_1, x_2) \in X \setminus \{(0, 0)\}, \\ 0, & (x_1, x_2) = (0, 0). \end{cases}$$

Then f is continuous on (X, \mathcal{T}) , since X is discrete, but f is not separately continuous on $X_1 \times X_2$ at the point $(0, 0)$ by definition 3.1. Therefore, it is natural to find conditions on a topology on a product of topological spaces such that the implication " f is continuous \Rightarrow f is separately continuous " holds.

Definition 3.6. Let $X \subseteq \prod_{t \in T} X_t$ be an \mathcal{S} -open subset of a product of topological spaces (X_t, \mathcal{T}_t) and let \mathcal{T} be a topology on X . We say that \mathcal{T} is *coordinated with the family* $(\mathcal{T}_t : t \in T)$ if

$$\lim_{x \rightarrow a_t} a_t^x = a \quad (3.1)$$

for all $t \in T$ and $a = (a_s)_{s \in T} \in X$.

Proposition 3.2. Let \mathcal{T} be a topology on an \mathcal{S} -open subset X of a product of topological spaces (X_t, \mathcal{T}_t) coordinated with the family $(\mathcal{T}_t : t \in T)$, Y is a (metric) space and $f : X \rightarrow Y$ be a (strongly separately) continuous mapping at a point $a \in X$. Then f is separately continuous at a .

Proof. Fix $t \in T$. Assume that f is continuous at the point a and denote $g(x) = f(a_t^x)$ for all $x \in X_t$. Then $\lim_{x \rightarrow a_t} g(x) = \lim_{x \rightarrow a_t} f(a_t^x) = f(a) = g(a_t)$.

We argue similarly in the case Y is a metric space and f is strongly separately continuous. \square

Proposition 3.3. *Let $X \subseteq \prod_{t \in T} X_t$ be an \mathcal{S} -open subset of a product of topological spaces (X_t, \mathcal{T}_t) and let \mathcal{T} be a topology on X . If one of the following conditions holds*

- (1) $(X, \mathcal{T}) = (X, \tau)$, or
 - (2) $(X_t, \mathcal{T}_t) = (\mathbb{R}, \mathcal{E})$ for every $t \in T$ and (X, \mathcal{T}) is a topological vector space,
- then \mathcal{T} is coordinated with $(\mathcal{T}_t : t \in T)$.

Proof. (1) It immediately implies from the definition of the Tychonoff topology.
 (2) Fix $a = (a_s)_{s \in T} \in X$ and $t \in T$. Without loss of generality we may assume that $a_s = 0$ for all $s \in S$. Let $b = 1$. Then $a_t^b \in \sigma_1(a) \subseteq X$. Since the function $\varphi : (X, \mathcal{T}) \times \mathbb{R} \rightarrow (X, \mathcal{T})$, $\varphi(y, \lambda) = \lambda y$, is continuous, we have that $\lim_{\lambda \rightarrow 0} \varphi(y, \lambda) = 0$ in (X, \mathcal{T}) for every $y \in X$. Then $\lim_{x \rightarrow 0} a_t^x = \lim_{x \rightarrow 0} \varphi(a_t^b, x) = 0$ in (X, \mathcal{T}) . \square

Theorem 3.4. *Let $X \subseteq \prod_{t \in T} X_t$ be an \mathcal{S} -open set and (Y, d) be a metric space. A mapping $f : (X, \mathcal{S}) \rightarrow Y$ is continuous if and only if $f : (X, \mathcal{T}) \rightarrow Y$ is strongly separately continuous for an arbitrary topology \mathcal{T} on X .*

Proof. Necessity. Fix a topology \mathcal{T} on X and consider the partition $(\sigma(x_i) : i \in I)$ of the set X on \mathcal{S} -components $\sigma(x_i)$. We notice that $f|_{\sigma(x_i)} = y_i$, where $y_i \in Y$ for all $i \in I$, since f is continuous on (X, \mathcal{S}) . Let $a = (a_t)_{t \in T} \in X$ and $t \in T$. If $x \in X$, then $x \in \sigma(x_i)$ for some $i \in I$. Moreover, $x_t^a \in \sigma(x_i)$. Then $f(x) = f(x_t^a) = y_i$. Hence, $d(f(x), f(x_t^a)) = 0$ for all $x \in X$. Hence, f is strongly separately continuous on (X, \mathcal{T}) .

Sufficiency. Put $\mathcal{T} = \mathcal{S}$. Fix $x_0 \in X$ and show that f is continuous at x_0 on (X, \mathcal{S}) . Let $x_0 \in \sigma(x_i)$ for some $i \in I$. Since f is strongly separately continuous at x_0 and $\sigma(x_0) = \sigma(x_i)$, we have $d(f(x), f(x_t^{x_0})) = 0$ for all $x \in \sigma(x_i)$ and $t \in T$. Consequently, $f(x) = f(x_0)$ for all $x \in \sigma(x_i)$. Since the set $\sigma(x_i)$ is open in (X, \mathcal{S}) , f is continuous at x_0 . \square

Now we give a necessary and sufficient condition for a strongly separately continuous mapping to be continuous.

Theorem 3.5. *Let $X \subseteq \prod_{t \in T} X_t$ be an \mathcal{S} -open subset of a product of topological spaces X_t , \mathcal{T} be a topology on X such that (X, \mathcal{T}) is a locally projectively symmetric space, (Y, d) be a metric space and let $f : (X, \mathcal{T}) \rightarrow Y$ be a strongly*

separately continuous mapping at the point $a = (a_t)_{t \in T} \in X$. Then f is continuous at the point a if and only if

$$\begin{aligned} &\text{for every } \varepsilon > 0 \text{ there exist a set } T_0 \subseteq T \text{ with } |T_0| < \aleph_0 \\ &\text{and a neighborhood } U \text{ of } a \text{ in } (X, \mathcal{T}) \text{ such that} \\ &d(f(a), f(x_{T_0}^a)) < \varepsilon \text{ for all } x \in U. \end{aligned} \quad (3.2)$$

Proof. NECESSITY. Suppose f is continuous at the point a and $\varepsilon > 0$. Take a neighborhood U of a such that $d(f(x), f(a)) < \varepsilon$ for all $x \in U$ and put $T_0 = \emptyset$. Then $x_{T_0}^a = x$, which implies condition (3.2).

SUFFICIENCY. Fix $\varepsilon > 0$. Using the condition of the theorem we take a finite set $T_0 \subseteq T$ and a neighborhood U of a in (X, τ) such that

$$d(f(a), f(x_{T_0}^a)) < \frac{\varepsilon}{2}$$

for every $x \in U$. If $T_0 = \emptyset$, then $d(f(x), f(a)) < \varepsilon$ for all $x \in U$. Now assume $T_0 = \{t_1, \dots, t_n\}$. Since f is strongly separately continuous at a , for every $k = 1, \dots, n$ we choose a neighborhood V_k of the point a such that

$$d(f(x), f(x_{t_k}^a)) < \frac{\varepsilon}{2n}$$

for all $x \in V_k$. We take a projectively symmetric with respect to the point a neighborhood W of a such that

$$W \subseteq U \cap \left(\bigcap_{k=1}^n V_k \right).$$

Since W is projectively symmetric set with respect to a , we can show inductively that $x_{\{t_1, \dots, t_k\}}^a \in W$ for every $k = 1, \dots, n$ and for every $x \in W$. Then for all $x \in W$ we have

$$\begin{aligned} d(f(x), f(a)) &\leq d(f(x), f(x_{T_0}^a)) + d(f(x_{T_0}^a), f(a)) < \\ &< d(f(x), f(x_{t_1}^a)) + d(f(x_{t_1}^a), f(x_{\{t_1, t_2\}}^a)) + \dots + \\ &\quad + d(f(x_{\{t_1, \dots, t_{n-1}\}}^a), f(x_{\{t_1, \dots, t_n\}}^a)) + \frac{\varepsilon}{2} < \\ &< \frac{\varepsilon}{2n} \cdot n + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, f is continuous at the point a . \square

We notice that the similar condition given by Visnyai in [3] for functions defined on ℓ_2 is stronger than condition (3.2).

The following corollary generalizes the result of Dzagnidze [1, Theorem 2.1].

Corollary 3.6. *Let X be an \mathcal{S} -open subset of a product $\prod_{t \in T} X_t$ of topological spaces X_t , $|T| < \aleph_0$ and (Y, d) be a metric space. Then any strongly separately continuous mapping $f : (X, \tau) \rightarrow Y$ is continuous.*

Proof. Fix an arbitrary point $a \in X$ and a strongly separately continuous mapping $f : (X, \tau) \rightarrow Y$. It is easy to see that f satisfy condition (3.2). Indeed, for $\varepsilon > 0$ we put $T_0 = T$ and $U = X$. Then for all $x \in U$ we have $x_{T_0}^a = a$ and consequently $d(f(a), f(x_{T_0}^a)) = 0 < \varepsilon$. Hence, f is continuous at a by Theorem 3.5, provided (X, τ) is a locally projectively symmetric space. \square

The proposition below shows that Corollary 3.6 is not valid for a product of infinitely many topological spaces.

Proposition 3.7. *Let $X = \prod_{t \in T} X_t$ be a product of topological spaces X_t , where $|X_t| > 1$ for every $t \in T$, let $|T| > \aleph_0$ and (Y, d) be a metric space with $|Y| > 1$. Then there exists a strongly separately continuous everywhere discontinuous mapping $f : (X, \tau) \rightarrow Y$.*

Proof. Fix $x_0 \in X$ and $y_1, y_2 \in Y$, $y_1 \neq y_2$. According to Proposition 2.1, $\sigma(x_0) \neq \emptyset \neq X \setminus \sigma(x_0)$. Set $f(x) = y_1$ if $x \in \sigma(x_0)$ and $f(x) = y_2$ if $x \in X \setminus \sigma(x_0)$. Then f is everywhere discontinuous on X . Indeed, let $a \in X$ and $f(a) = y_1$. Take an open neighborhood V of y_1 such that $y_2 \notin V$. If U is an arbitrary neighborhood of a in (X, τ) , then there is $x \in U \setminus \sigma(x_0)$. Therefore, $f(x) = y_2 \notin V$ and f is discontinuous at a . Similarly, f is discontinuous at a in the case $f(a) = y_2$.

Since the set $\sigma(x_0)$ is clopen in (X, \mathcal{S}) , the mapping $f : (X, \mathcal{S}) \rightarrow Y$ is continuous. It remains to apply Theorem 3.4. \square

4. A NECESSARY CONDITION ON THE DISCONTINUITY POINT SET

Let $f : X \rightarrow Y$ a mapping between topological spaces. By $C(f)$ we denote the continuity point set of f and by $D(f)$ we denote the discontinuity point set of f .

Let \mathcal{U}_x be a system of all neighborhoods of a point x in X . For a mapping $f : X \rightarrow (Y, d)$ we set

$$\omega_f(A) = \sup_{x', x'' \in A} d(f(x'), f(x'')) \quad \text{and} \quad \omega_f(x) = \inf_{U \in \mathcal{U}_x} \omega_f(U).$$

Definition 4.1. A set $W \subseteq \sigma(a)$ is called *nearly open in $(\sigma(a), \tau)$* if for any finite set $T_0 \subseteq T$ the set

$$W_{T_0} = \{z \in \prod_{t \in T_0} X_t : a_{T_0}^z \in W\}$$

is open in the space $(\prod_{t \in T_0} X_t, \tau)$.

Theorem 4.1. *Let $(X_t : t \in T)$ be a family of topological spaces, $X = \prod_{t \in T} X_t$, $a = (a_t)_{t \in T} \in X$, (Y, d) be a metric space and $f : (\sigma(a), \tau) \rightarrow Y$ be a strongly*

separately continuous mapping. Then the discontinuity point set $D(f)$ of f is nearly open in $(\sigma(a), \tau)$.

Proof. Let $T_0 \subseteq T$ be an arbitrary finite set and $Z = \prod_{t \in T_0} X_t$. For $z \in Z$ we write $\varphi(z) = a_{T_0}^z$ and $G = (D(f))_{T_0}$.

If $T_0 = \emptyset$, then $G = \emptyset$. Now let $T_0 = \{t_1, \dots, t_n\}$, $w = (w_t)_{t \in T_0} \in G$, $u = \varphi(w) \in \sigma(a)$ and $\varepsilon = \frac{1}{3}\omega_f(u)$. We observe that $\varepsilon > 0$, provided f is discontinuous at u . Since f is strongly separately continuous at the point u , there exists a basic neighborhood U_0 of u in (X, τ) such that

$$d(f(x), f(x_{\{t\}}^u)) < \frac{\varepsilon}{6n} \quad (4.1)$$

for all $t \in T_0$ and $x \in U_0 \cap \sigma(a)$. Since the mapping $\varphi : (Z, \tau) \rightarrow (\sigma(a), \tau)$ is continuous, there exists a basic neighborhood W_0 of w in (Z, τ) such that $\varphi(W_0) \subseteq U_0$.

We show that

$$d(f(x), f(x_{T_0}^{\varphi(z)})) < \frac{\varepsilon}{3} \quad (4.2)$$

for any $z \in W_0$ and $x \in U_0 \cap \sigma(a)$. Let $x_0 = x$ and $x_k = (x_{k-1})_{t_k}^{\varphi(z)}$, $k = 1, \dots, n$. Then $x_n = x_{T_0}^{\varphi(z)}$. Moreover, since $x, \varphi(z) \in U_0$, $x_k \in U_0$ for every k . It follows from (4.1) that

$$d(f(x_{k-1}), f((x_{k-1})_{t_k}^u)) < \frac{\varepsilon}{6n} \quad \text{i} \quad d(f(x_k), f((x_k)_{t_k}^u)) < \frac{\varepsilon}{6n}$$

for every $k = 1, \dots, n$. Taking into account the equality $(x_{k-1})_{t_k}^u = (x_k)_{t_k}^u$, we obtain

$$d(f(x_{k-1}), f(x_k)) < \frac{\varepsilon}{3n}.$$

Hence,

$$d(f(x), f(x_{T_0}^{\varphi(z)})) = d(f(x_0), f(x_n)) \leq \sum_{k=1}^n d(f(x_{k-1}), f(x_k)) < \frac{\varepsilon}{3}.$$

Now we prove that

$$\omega_f(\varphi(z)) \geq \frac{\varepsilon}{3}$$

for all $z \in W_0$. Let $z \in W_0$ and $x = \varphi(z)$. Since $\omega_f(u) = 3\varepsilon$, there exists a net $(u_\lambda)_{\lambda \in \Lambda}$ of points of $\sigma(a) \cap U_0$ such that $u_\lambda \xrightarrow{\lambda \in \Lambda} u$ in (X, τ) and $d(f(u_\lambda), f(u)) \geq \varepsilon$ for every $\lambda \in \Lambda$. We notice that $u_{T_0}^x = x$. Therefore, $v_\lambda = (u_\lambda)_{T_0}^x \xrightarrow{\lambda \in \Lambda} x$. It follows from (4.2) that

$$d(f(u), f(x)) < \frac{\varepsilon}{3} \quad \text{i} \quad d(f(u_\lambda), f(v_\lambda)) < \frac{\varepsilon}{3}.$$

Hence,

$$d(f(x), f(v_\lambda)) \geq d(f(u), f(u_\lambda)) - d(f(u), f(x)) - d(f(u_\lambda), f(v_\lambda)) > \frac{\varepsilon}{3},$$

consequently, $\omega_f(x) \geq \frac{\varepsilon}{3}$. Therefore, $x \in D(f)$. Thus, $W_0 \subseteq G$, which implies that G is open in (Z, τ) . \square

5. A SUFFICIENT CONDITION ON THE DISCONTINUITY POINT SET

The definition of strongly separately continuous mapping easily implies the following properties.

Proposition 5.1. *Let X be an \mathcal{S} -open subset of a product $\prod_{t \in T} X_t$ of a family of topological spaces X_t and \mathcal{T} be a topology on X .*

If $f, g : (X, \mathcal{T}) \rightarrow \mathbb{R}$ are strongly separately continuous mappings at $x_0 \in X$, then the mappings $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $|f(x)|$, $\min\{f(x), g(x)\}$, $\max\{f(x), g(x)\}$ are strongly separately continuous at x_0 .

If $f(x) = \sum_{n=1}^{\infty} f_n(x)$ is a sum of a uniformly convergent series of strongly separately continuous at $x_0 \in X$ mappings $f_n : (X, \mathcal{T}) \rightarrow \mathbb{R}$, then f is strongly separately continuous at x_0 .

If $(X, \|\cdot\|)$ is a normed space, $a \in X$ and $r > 0$, then we write

$$B(a, r) = \{x \in X : \|x - a\| < r\} \quad \text{and} \quad B[a, r] = \{x \in X : \|x - a\| \leq r\}.$$

Theorem 5.2. *Let $((X_n, \|\cdot\|_n))_{n=1}^{\infty}$ be a sequence of normed spaces, $a \in \prod_{n=1}^{\infty} X_n$, $w = (w_n)_{n=1}^{\infty} \in \sigma(a)$, $(r_n)_{n=1}^{\infty}$ be a sequence of positive reals and*

$$W = \left(\prod_{n=1}^{\infty} B(w_n, r_n) \right) \cap \sigma(a).$$

Then there exists a strongly separately continuous function $f : (\sigma(a), \tau) \rightarrow [0, 1]$ such that

$$W = D(f) \subseteq f^{-1}(0).$$

Proof. Assume without loss of generality that $a = (0, 0, \dots)$. We put $X = (\sigma(a), \tau)$ and for every $n \in \mathbb{N}$ let

$$B_n = B(w_n, r_n), \quad Y_n = X_1 \times \dots \times X_n \quad \text{and} \\ d_n(x, y) = \max_{1 \leq i \leq n} \|x_i - y_i\|_i \quad \text{for all } x, y \in Y_n.$$

If $x = (x_n)_{n=1}^{\infty} \in X$ and $k \in \mathbb{N}$, then $p_k(x)$ stands for (x_1, \dots, x_k) .

For every $x = (x_n)_{n=1}^{\infty} \in X$ we set $h(x) = \left(\frac{1}{r_n}(x_n - w_n) \right)_{n=1}^{\infty}$. Then $h : X \rightarrow X$ is a homeomorphism.

For every $n \in \mathbb{N}$ define

$$\begin{aligned} B'_n &= \{x \in X_n : \|x\|_n < 1\}, \quad S_n = \{x \in X_n : \|x\|_n = 1\}, \\ A_1 &= X_1 \setminus B'_1, \quad A_n = \prod_{i=1}^{n-1} B'_i \times (X_n \setminus B'_n) \text{ if } n \geq 2, \\ C_n &= (A_n \times \prod_{i=n+1}^{\infty} X_i) \cap X, \quad C = \bigsqcup_{n=1}^{\infty} C_n. \end{aligned}$$

We observe that

$$W' = h(W) = \left(\prod_{n=1}^{\infty} B'_n \right) \cap X \quad \text{and} \quad C = X \setminus W'.$$

Now for every $n \in \mathbb{N}$ we consider a function $f_n : X \rightarrow \mathbb{R}$,

$$f_n(x) = d_n(p_n(x), Y_n \setminus A_n),$$

and for all $x \in X$ set

$$g(x) = \begin{cases} f_n(x), & \text{if } p_n(x) \in A_n \text{ for some } n \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

Claim 1. The function $g : X \rightarrow \mathbb{R}$ is strongly separately continuous.

Proof. Fix $u \in X$ and $k \in \mathbb{N}$. For $x \in X$ we write $y = x_k^u$ and estimate the difference $|g(x) - g(y)|$. If $p_n(x) \in A_n$ and $p_m(y) \in A_m$ for some $n, m \in \mathbb{N}$, then, using the equality $f_n(y) = f_m(x) = 0$ in the case $n \neq m$, we obtain

$$\begin{aligned} |g(x) - g(y)| &\leq |f_n(x) - f_n(y)| + |f_m(x) - f_m(y)| \leq \\ &\leq |d_n(p_n(x), p_n(y))| + |d_m(p_m(x), p_m(y))| \leq 2\|x_k - u_k\|_k. \end{aligned}$$

If $p_n(x) \in A_n$ for some n and $y \notin C$, then $g(y) = f_n(y) = 0$ and

$$|g(x) - g(y)| \leq |d_n(p_n(x), p_n(y))| \leq \|x_k - u_k\|_k.$$

The same estimation is valid if $x \notin C$ and $y \in C$. Finally, if $x, y \notin C$, then $|g(x) - g(y)| = 0$. Hence, for all $x \in X$ we have

$$|g(x) - g(y)| \leq 2\|x_k - u_k\|_k,$$

which implies that

$$0 \leq \lim_{x \rightarrow u} |g(x) - g(y)| \leq 2 \lim_{x \rightarrow u} \|x_k - u_k\|_k = 0.$$

Therefore, g is strongly separately continuous at u . □

Claim 2. The equality

$$d_n(u, Y_n \setminus A_n) = \min_{1 \leq i \leq n} \|u_i - S_i\|_i. \quad (5.1)$$

holds for all $n \in \mathbb{N}$ and $u = (u_1, \dots, u_n) \in A_n$.

Proof. Fix $n \in \mathbb{N}$, $u = (u_1, \dots, u_n) \in A_n$ and let $B = Y_n \setminus A_n$, $\rho_i = \|u_i - S_i\|_i$ if $i = 1, \dots, n$, and $\rho = \min_{1 \leq i \leq n} \rho_i$.

We show that $d_n(u, B) \geq \rho$. If $\|u_n\|_n = 1$, then $\rho_n = 0 = \rho$ and the inequality is obvious. Suppose $\|u_n\|_n > 1$, choose any $y = (y_1, \dots, y_n)$ with $d_n(u, y) < \rho$ and check that $y \in A_n$. Notice that $\|u_i - y_i\|_i < \rho_i$ for every $i = 1, \dots, n$. Assume that there exists $1 \leq i \leq n-1$ such that $\|y_i\|_i \geq 1$. Then there exists $b \in S_i$ such that $b = u_i + t(y_i - u_i)$ for some $t \in (0, 1]$. Then $\|u_i - b\|_i = t\|u_i - y_i\|_i < \rho_i$, a contradiction. Hence, $y_i \in B'_i$ for every $i = 1, \dots, n-1$. Similarly we can show that $y_n \notin B'_n$. Hence, $y \in A_n$.

Now we show that $d_n(u, B) \leq \rho$. Fix $\varepsilon > 0$ and assume that $\rho = \rho_i$ for some $1 \leq i \leq n$. If $i < n$, then there exists $v_i \in S_i$ with $\|u_i - v_i\|_i < \rho + \varepsilon$. If $i = n$, then, since $\|u_n - S_n\|_n = \|u_n - B'_n\|_n$, there exists $v_n \in B'_n$ such that $\|u_n - v_n\|_n < \rho + \varepsilon$. Let $v_j = u_j$ for $j \neq i$. Then $v = (v_1, \dots, v_n) \in B$ and $d_n(u, v) < \rho + \varepsilon$, which implies that $d_n(u, B) \leq \rho$.

Therefore, $d_n(u, B) = \rho$. \square

Claim 3. $W' \subseteq D(g)$.

Proof. Fix $u = (u_1, \dots, u_n, 0, \dots) \in W'$. For every $i = 1, \dots, n$ we set $\rho_i = \|u_i - S_i\|_i$ and $\rho = \min_{1 \leq i \leq n} \rho_i$. Then $\rho \in (0, 1]$. For every $i \geq n+1$ we choose $x_i \in X_i$ such that $\|x_i\|_i = 1 + \rho$ and consider a sequence $(x^m)_{m=1}^\infty$ such that

$$x^m = (u_1, \dots, u_n, 0, \dots, 0, x_{m+n}, 0, \dots).$$

Clearly, $x^m \in C_{m+n}$ and $x^m \rightarrow u$. Since

$$\|x_{m+n} - z\|_{m+n} \geq |\|x_{m+n}\|_{m+n} - \|z\|_{m+n}| = \rho$$

for all $z \in S_{m+n}$,

$$g(x^m) = \min\{\rho, \|x_{m+n} - S_{m+n}\|_{m+n}\} = \rho.$$

Hence, $\lim_{m \rightarrow \infty} g(x^m) = \rho > 0 = g(u)$, which implies the discontinuity of g at u . \square

Claim 4. $C \subseteq C(g)$.

Proof. Fix $u = (u_n)_{n=1}^\infty \in C$ and $\varepsilon > 0$. Let $p_n(u) \in A_n$ for some $n \in \mathbb{N}$ and consider the case $\|u_n\|_n > 1$. Since $\psi : Y_n \rightarrow \mathbb{R}$, $\psi(x_1, \dots, x_n) = \min_{1 \leq i \leq n} \|x_i - S_i\|_i$, is continuous at $p = p_n(u)$, there exists a neighborhood $U = U_1 \times \dots \times U_n$ of p in Y_n such that $|\psi(x) - \psi(u)| < \varepsilon$ for all $x \in U$. Let

$$G = \prod_{i=1}^{n-1} (U_i \cap B'_i) \times (U_n \cap (X_n \setminus \overline{B'_n})) \times \prod_{i=n+1}^\infty X_i.$$

Then $|g(x) - g(u)| = |\psi(x) - \psi(u)| < \varepsilon$ for all $x \in G \cap X$.

Now suppose $\|u_n\|_n = 1$. Then $g(u) = f_n(u) = 0$. Set

$$V = \prod_{i=1}^{n-1} B'_i \times B(u_n, \varepsilon) \times \prod_{i=n+1}^{\infty} X_i.$$

Let $x \in V \cap X$. If $\|x_n\|_n \leq 1$, then $g(x) = 0$, and if $\|x_n\|_n > 1$, then

$$g(x) = \min_{1 \leq i \leq n} \|x_i - S_i\|_i \leq \|x_n - S_n\|_n \leq \|x_n - u_n\|_n < \varepsilon.$$

Hence, $|g(x) - g(u)| < \varepsilon$ for all $x \in V \cap X$.

Therefore, $u \in C(g)$. □

Claim 3 and Claim 4 imply that $D(g) = W'$. Moreover, $g(x) = 0$ for all $x \in W'$.

Consider the function $\varphi : X \rightarrow \mathbb{R}$ such that $\varphi(x) = g(h(x))$ for all $x \in X$.

Fix $n \in \mathbb{N}$ and $u \in X$. Then $h(x_n^u) = (h(x))_n^{h(u)}$ for every $x \in X$. We have

$$\begin{aligned} \lim_{x \rightarrow u} |\varphi(x) - \varphi(x_n^u)| &= \lim_{x \rightarrow u} |g(h(x)) - g(h(x_n^u))| = \\ &= \lim_{h(x) \rightarrow h(u)} |g(h(x)) - g((h(x))_n^{h(u)})| = 0, \end{aligned}$$

since g is strongly separately continuous at $h(u)$ with respect to the n -th variable. Hence, $\varphi : X \rightarrow \mathbb{R}$ is strongly separately continuous.

If is easy to see that φ is continuous at $x \in X$ if and only if g is continuous at $h(x) \in X$. Hence, $D(\varphi) = h^{-1}(D(g)) = h^{-1}(W') = W$. Moreover, $\varphi^{-1}(0) = h^{-1}(g^{-1}(0)) \supseteq h^{-1}(W') = W$.

Finally, for every $x \in X$ we put

$$f(x) = \min\{\varphi(x), 1\}.$$

Note that $D(f) = D(\varphi) = W$ and $f(x) = 0$ for all $x \in W$.

It remains to observe that $f : X \rightarrow [0, 1]$ is strongly separately continuous by Proposition 5.1. □

Theorem 5.3. *Let $((X_n, \|\cdot\|_n))_{n=1}^{\infty}$ be a sequence of finite-dimensional normed spaces, $a \in \prod_{n=1}^{\infty} X_n$ and $W \subseteq \sigma(a)$ be a nearly open set. Then there exists a strongly separately continuous function $f : (\sigma(a), \tau) \rightarrow [0, 1]$ such that $D(f) = W$.*

Proof. Without loss of generality we can assume that $a = (0, 0, \dots)$. For every $n \in \mathbb{N}$ let

$$\begin{aligned} Y_n &= X_1 \times \dots \times X_n, \quad Z_n = Y_n \times \{0\} \times \{0\} \times \dots, \\ G_n &= \{(x_1, \dots, x_n) \in Y_n : (x_1, \dots, x_n, 0, \dots) \in W\}, \end{aligned}$$

and $X = (\sigma(0), \tau)$.

Let $x = (x_n)_{n=1}^\infty$ be an arbitrary point of W such that $x_n = 0$ for all $n > N$. Since W is nearly open, for every $k = 1, \dots, N$ there exists $r_k(x) > 0$ such that $F_1 = \prod_{k=1}^N B[x_k, r_k(x)] \subseteq G_N$. Since compact set $K_1 = F_1 \times \{0\}$ is contained in the open in Y_{N+1} set G_{N+1} , we can find $r_{N+1}(x) > 0$ such that

$$F_2 = F_1 \times B[0, r_{N+1}(x)] \subseteq G_{N+1}.$$

Now the compactness of $K_2 = F_2 \times \{0\} \subseteq W|_{Y_{N+2}}$ implies that there exists $r_{N+2}(x) > 0$ such that

$$F_3 = F_2 \times B[0, r_{N+2}(x)] \subseteq G_{N+2}.$$

By repeating this process, we obtain a sequence $(r_n(x))_{n=1}^\infty$ of positive reals such that

$$\left(\prod_{n=1}^\infty B[x_n, r_n(x)] \right) \cap X \subseteq W.$$

Now let $W(x) = \left(\prod_{n=1}^\infty B(x_n, r_n(x)) \right) \cap X$.

Hence, $W = \bigcup_{x \in W} W(x)$. Since for every $n \in \mathbb{N}$ the family $(W(x) \cap Z_n : x \in W)$ forms an open covering of $V_n = W \cap Z_n$ in Z_n , there exists a countable set $I_n \subseteq W$ such that the family $(W(x) \cap Z_n : x \in I_n)$ is a covering of V_n . Put $I = \bigcup_{n=1}^\infty I_n$.

Then

$$W = \bigcup_{n=1}^\infty V_n = \bigcup_{n=1}^\infty \bigcup_{x \in I_n} (W(x) \cap Z_n) = \bigcup_{x \in I} W(x).$$

Let $I = \{x_m : m \in \mathbb{N}\}$ and $W_m = W(x_m)$.

According to Theorem 5.2 there exists a sequence $(f_m)_{m=1}^\infty$ of strongly continuous functions $f_m : X \rightarrow [0, 1]$ such that $D(f_m) = W_m \subseteq f_m^{-1}(0)$. The last inclusion implies that every f_m is lower semi-continuous on X . For all $x \in X$ we define

$$f(x) = \sum_{m=1}^\infty \frac{1}{2^m} f_m(x).$$

Then Proposition 5.1 implies that $f : X \rightarrow [0, 1]$ is strongly separately continuous function. Moreover, f is lower semi-continuous as a sum of uniformly convergent series of lower semi-continuous functions.

Taking into account that $D(g_1 + g_2) = D(g_1) \cup D(g_2)$ for any two lower semi-continuous functions g_1 and g_2 , we obtain

$$D(f) = \bigcup_{m=1}^\infty D(f_m) = W.$$

□

Combining Theorems 4.1 and 5.3 we obtain

Theorem 5.4. *Let $X = \prod_{n=1}^{\infty} X_n$ be a product of finite-dimensional normed spaces X_n , $a \in X$ and $W \subseteq \sigma(a)$. Then W is the discontinuity point set of some strongly separately continuous function $f : (\sigma(a), \tau) \rightarrow \mathbb{R}$ if and only if W is nearly open in $(\sigma(a), \tau)$.*

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